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COMMUTATIVE TORSION STABLE RINGS

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Introduction

All rings in our paper are assumed to be commutative with identity. A localizing subcategory (hereditary torsion class) is called *stable* if it is closed under taking injective envelopes. A ring R is called stable if every localizing subcategory of Mod-R is stable. This is equivalent to the following: For any R-module M and for any injective R-module E the relation $\mathrm{Hom}(M, E)=0$ implies $\mathrm{Hom}(E(M), E)=0$ where $E(M)$ is the injective envelope of M. Examples of such a ring include all commutative rings which are either noetherian or perfect.

It has been shown in [1] and [8] that stable rings whether in a commutative or noncommutative setting have properties similar to commutative noetherian rings. In this paper we present more evidence on the closeness of commutative stable rings to commutative noetherian rings. We also give numerous examples of stable rings.

In Section 1 we give the basic results on stable rings which are of interest by themselves as well as useful later on. In Section 2 we present some examples. Previously, the only known examples of stable rings were either noetherian or perfect. We give families of stable rings which are neither. In Section 3 we prove that stability plus various other assumptions imply a ring is noetherian. As corollaries we are able to show that certain rings of the form $K+P$, where P is a maximal ideal of a K algebra, have infinite global dimension and are not coherent.

$E_R(M)$ (or $E(M)$ when there is no ambiguity) will denote the injective envelope of an R-module M, and Mod-R is used to denote the category of R-modules. Given an ideal I of R and an R-module M, $\mathrm{ann}_M I$ is the largest submodule of M which is killed by I. Finally given a module M and a prime ideal P, M_P is the localization of M at P.

1. Basic results

In this section we present basic results about commutative stable rings that will

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be used throughout the paper. These results, which are well known when R is noetherian (see [6] and [10] for instance), also highlight the fact that stable rings are a generalization of noetherian rings.

Proposition 1.1. *If R is a stable ring, then so is every homomorphic image of R.*

Proof. Let $\bar{R}=R/I$ be a factor ring of R. Assume $\mathrm{Hom}_{\bar{R}}(M,E)=0$ where E is an injective $\bar{R}$-module and M is an arbitrary $\bar{R}$-module. Clearly $\mathrm{Hom}_R(M,E)=0$. Recall that for any $\bar{R}$-module X, $E_{\bar{R}}(X)=\{e\in E_R(X)\mid eI=0\}$. Thus $\mathrm{Hom}_R(M,E_R(E))=0$, since any homomorphic image of M is killed by I. So by the stability of R, $\mathrm{Hom}_R(E_R(M),E_R(E))=0$. Since $E_{\bar{R}}(M)\subseteq E_R(M)$ we have the proposition. □

The *Gabriel dimension* will be defined in a categorical setting and the Gabriel dimension of a ring R is the Gabriel dimension of the category Mod-R. Let $\mathscr{C}$ be a Grothendieck category. $\mathscr{D}_0$ is the smallest localizing subcategory of $\mathscr{C}$ which is generated by the objects of $\mathscr{C}$ with finite length; $\mathscr{D}_1$ is the smallest localizing subcategory of $\mathscr{C}$ containing those objects $M\in\mathscr{C}$ that have finite length in the quotient category $\mathscr{C}/\mathscr{D}_0$, etc. We get a transfinite filtration $\mathscr{D}_0\subseteq\mathscr{D}_1\subseteq\cdots\subseteq\mathscr{D}\subseteq\cdots$, the *Gabriel filtration*, where $\mathscr{D}_\alpha$, for a limit ordinal α, is defined as usual. An object $M\in\mathscr{D}_\beta$ for some β has Gabriel dimension and G-dim$(M)=\alpha$ if $\alpha=\min\{\beta\mid M\in\mathscr{D}_\beta\}$ otherwise M does not have Gabriel dimension. The Grothendieck category $\mathscr{C}$ has Gabriel dimension and G-dim$(\mathscr{C})=\gamma$ if $\mathscr{C}=\bigcup\{\mathscr{D}_\alpha\mid\alpha\le\gamma\}$ and γ is the smallest ordinal with this property. If an R-module M is noetherian, then it has Gabriel dimension. For details and proofs see [4]. Note that their definition is slightly different than ours since they let $\mathscr{D}_0=\{0\}$. However, for infinite ordinals the definitions agree.

We say a ring R is seminoetherian if it has Gabriel dimension. Such rings have two properties that are useful to us. First, given a nonzero R-module M, $\mathrm{ass}_R(M)$, (or ass(M) when there is no ambiguity) which denotes the set of prime ideals of R associated to M, is not empty if R is seminoetherian. Second, to check stability in a seminoetherian ring we only have to check some of the localizing subcategories of Mod-R. We will go into more detail on this later. It turns out that all the examples of stable rings we give in the paper are in fact seminoetherian. However, we don't know if this is by necessity.

Given an R-module M, $S(M)$ will be the sum of all submodules of M which are isomorphic to R/P for some prime ideal P (clearly $P\in\mathrm{ass}(M)$). If R is seminoetherian, $S(M)$ is essential in M.

Proposition 1.2. *Let R be a seminoetherian stable ring and let M and N be R-modules. Then* $\mathrm{Hom}_R(M,N)\neq 0$ *if and only if there exists* $Q\in\mathrm{ass}(M)$ *and* $P\in\mathrm{ass}(N)$ *such that* $Q\subseteq P$.

Proof. The 'if' direction is clear. Conversely, assume $\mathrm{Hom}_R(M,N)\neq 0$. Then $\mathrm{Hom}_R(S(M),N)\neq 0$ since R is seminoetherian and stable. Thus $\mathrm{Hom}_R(R/Q,N)\neq 0$ for some $Q\in \mathrm{ass}(M)$. Let $f:R/Q\to N$ be nonzero. There exists $P\in \mathrm{ass}(f(R/Q)\subseteq \mathrm{ass}(N)$. Clearly $Q\subseteq P$. $\square$

For an R-module M we define the support of M, denoted $\mathrm{supp}(M)$, to be the set of prime ideals P in R such that $M_P\neq 0$. For any nonzero module M, $\mathrm{supp}(M)$ is not empty. In a noetherian ring it is well known that the minimal elements of $\mathrm{supp}(M)$ are in $\mathrm{ass}(M)$.

Corollary 1.3. *Let M be a module over a seminoetherian stable ring R. Then the minimal elements of* $\mathrm{supp}(M)$ *are in* $\mathrm{ass}(M)$.

Proof. Clearly $\mathrm{ass}(M)\subseteq \mathrm{supp}(M)$. Let P be a minimal element of $\mathrm{supp}(M)$. Then $\mathrm{Hom}(M,E(R/P))\neq 0$. So by Proposition 1.2 there exists $Q\in \mathrm{ass}(M)$ *with* $Q\subseteq P$. Since $Q\subseteq \mathrm{supp}(M)$ we have $Q=P$. $\square$

In a noetherian ring the injective hull of a simple module is artinian. This is not true for arbitrary stable rings (take any local perfect ring which is not noetherian). However we do have a generalization. A module M is called *semiartinian* if every nonzero homomorphic image of M has a nonzero socle.

Let M be an R-module. The socle series of M is defined recursively as follows:

$$\mathrm{soc}_1(M)=\mathrm{soc}(M),$$

$$\mathrm{soc}_{\alpha+1}(M)/\mathrm{soc}_\alpha(M)=\mathrm{soc}(M/\mathrm{soc}_\alpha(M)),$$

$$\mathrm{soc}_\alpha(M)=\sum_{\beta<\alpha}\mathrm{soc}_\beta(M)\quad\text{when } \alpha \text{ is a limit ordinal,}$$

$$\overline{\mathrm{soc}}(M)=\mathrm{soc}_\alpha(M)\quad\text{where } \alpha \text{ is the first ordinal for which}$$

$$\mathrm{soc}_\alpha(M)=\mathrm{soc}_{\alpha+1}(M).$$

Thus $\mathrm{soc}(M/\overline{\mathrm{soc}}(M))=0$. So M is semiartinian if and only if $M=\mathrm{soc}_\alpha(M)$ for some ordinal. Also note that being semiartinian is equivalent to having Gabriel dimension zero.

Proposition 1.4. *Let R be a stable ring, and let M be a simple R-module. Then $E(M)$ is semiartinian.*

Proof. Let H be a homomorphic image of $E(M)$. Since $\mathrm{Hom}(E(M),H)\neq 0$, it follows from the stability of R that $\mathrm{Hom}(M,H)\neq 0$. So M is isomorphic to a submodule of H. $\square$

It is well known that every ideal in a noetherian ring satisfies the AR property (in fact it satisfies the stronger Artin–Rees property). Namely, given an ideal I, a finitely generated module M and a submodule N, there exists a positive integer k such that $MI^k \cap N \subseteq NI$. We will generalize this result to stable rings by the use of localizing subcategories. Let I be a finitely generated ideal of some ring R. Let $\mathscr{T} = \{M \in \text{Mod-}R \mid \text{every element of } M \text{ is annnihilated by some power of } I\}$. Then $\mathscr{T}$ is a localizing subcategory of Mod-R. Given a module N and a submodule H we say H is $\mathscr{T}$-dense (or merely dense) if $N/H \in \mathscr{T}$. If N is finitely generated, then $H \subseteq N$ is dense if and only if $NI^k \subseteq H$ for some integer k. For an arbitrary module N and submodule k it is not true in general that dense submodules of K come from dense submodules of N, i.e., a dense submodule of K may not be the intersection of a dense submodule of N with K. However, if $\mathscr{T}$ is closed under essential extensions (i.e. $\mathscr{T}$ is stable), then every dense submodule of K is of the form $H \cap K$ where H is dense in N [3 11.4]. Thus we have proven the following.

Proposition 1.5. *Let R be a stable ring and let I be a finitely generated ideal of R. Let N be a finitely generated R module, and let K be a submodule of N. Then there exists an integer r such that $NI^r \cap K \subseteq KI$.*

We should point out that all the results of this section work for noncommutative rings as well. The only substitution needed is "associated prime torsion theory" for "associated prime ideal".

2. Examples

Previously, the only examples of stable rings to appear in the literature are either noetherian or perfect. In this section, using split extensions and subidealizer rings, we are able to give stable rings which are neither.

Let R be a ring and M an R-module. Then $R \ltimes M$ will denote the split extension of R by M. Recall that the abelian group structure of this ring is the same as $R \times M$, while multiplication is defined by $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$, for $r_i \in R$ and $m_i \in M$.

Theorem 2.1. *Let R be a stable seminoetherian ring and let M be an R-module. If M has the property that* $\operatorname{Hom}(M, E)$ *is isomorphic to a submodule of $\bigoplus E$ for every injective R-module E, then $R \ltimes M$ is stable.*

Proof. Denote $R \ltimes M$ by T. Let N be a T-module and let H be those elements of N killed by $0 \ltimes M$. Since $0 \ltimes M$ is nilpotent H is essential in N. Thus $E_T(H) = E_T(N)$. We claim that $E_T(H)$ is isomorphic to the module $\bar{E}$ where $\bar{E}$ is defined in the following manner: The abelian group structure of $\bar{E}$ is the same as $E_R(H) \times \operatorname{Hom}(M, E_R(H))$, while the T-module structure is given by $(r, m)(e, f) = (re + f(m), rf)$ for $(r, m) \in T$ and $(e, f) \in \bar{E}$. In general, given an injective R-module E we will denote the T-module obtained in this manner by

$E \ltimes \mathrm{Hom}(M, E)$. To prove the claim observe that the R-map $H \to \bar{E}$, defined by $h \to (h, 0)$, is also a T-homomorphism since H is annihilated by $0 \ltimes M$.

Also note that $\bar{E} \simeq \mathrm{Hom}_R(T, E_R(H))$ as T-modules. Thus $\bar{E}$ is injective over T. Finally, identifying H with its image in $\bar{E}$ we have to show $\bar{E}$ is an essential extension of H. Let $0 \neq (e, f) \in \bar{E}$. If $f = 0$, then $(r, 0)(e, 0) = (re, 0) \in H$ for some $r \in R$ where $re \neq 0$. If $f \neq 0$, then there exists $m \in M$ such that $f(m) \neq 0$. So $(0, m)(e, f) = (f(m), 0)$ and we are back to the first case.

Now let E be an injective T-module and let N be an arbitrary T-module. Assume $\mathrm{Hom}_T(N, E) = 0$. Then $\mathrm{Hom}_T(H, E) = 0$ where H is those elements of N killed by $0 \ltimes M$. We know $E \simeq E_1 \ltimes \mathrm{Hom}_R(M, E_1)$ and $E_T(H) \simeq E_2 \ltimes \mathrm{Hom}_R(M, E_2)$ where E_1 and E_2 are injective R-modules. Also note E_1 and E_2 are T-modules where $(0 \ltimes M)E_i = 0$. Clearly $\mathrm{Hom}_R(H, E_1) = 0$, for any R-homomorphism between H and E_1 is also a T-homomorphism. Thus $\mathrm{Hom}_R(E_2, E_1) = 0$, since R is stable and $E_2 = E_R(H)$. By hypothesis and the previous paragraph $E_T(H) \subseteq \bigoplus_I E_2$ and $E \subseteq \bigoplus_J E_1$ as R-modules. Thus $\mathrm{Hom}_R(E_T(H), E) = 0$ which of course implies $\mathrm{Hom}_T(E_T(H), E) = 0$. Therefore T is stable. □

Example 1. Let R be a noetherian ring which is not perfect. Let M be a semisimple module which contains only finitely many different isomorphism classes of simples. Given any injective module E, $\mathrm{Hom}(M, E)$ is either zero or semisimple. In either case $\mathrm{Hom}(M, E) \subseteq \bigoplus E$. Thus, by Theorem 2.1, $R \ltimes M$ is stable. However, it is not perfect and if M is infinitely generated, $R \ltimes M$ is not noetherian.

Let T be a ring and let A be an ideal of T. Recall that a subring S of T is called a *subidealizer* of A in T if S contains A. In this situation note that given any S-module N, NA is a T-module. We can deduce from [9, Theorem 1.1, Proposition 1.3(iv)] the following: Let P be a prime ideal of T such that $A \not\subseteq P$ and let $Q = P \cap S$. Then S_Q is naturally isomorphic to T_P as rings. Furthermore for any T-module M, M_P is isomorphic to M_Q as modules over this common ring of quotients. Thus, if E is an injective T-module which is P torsion free, $E_P = E$ is an injective module over $T_P = S_Q$. So it follows from [3, Corollary 6.8] that E is injective over S. Finally, note that this argument can easily be generalized to an injective T-module which is torsion free over any finite set of prime ideals $\{P_1, \ldots, P_n\}$ in T where $A \not\subseteq P_i$ for all $i = 1, 2, \ldots, n$.

Before we get to Theorem 2.4, which will give us more examples using subidealizers, we need a couple of preliminary results.

Lemma 2.3. *Let S be a subidealizer of A in T. Let M and N be T-modules such that* $\mathrm{ann}_N A = 0$. *Then any S-homomorphism from M to N is a T-homomorphism.*

Proof. Let $m \in M$ and let $f \in \mathrm{Hom}_S(M, N)$. If $t \in T$, then $f(m)t - f(mt)$ is killed by A since $tA \subseteq S$. Thus $f(m)t = f(mt)$. □

We also need a criteria for a seminoetherian R not to be stable. Any localizing subcategory $\mathcal{T}$ of Mod-R is cogenerated by some injective module E in the sense that $\mathcal{T}=\{M \text{ in Mod-}R \mid \mathrm{Hom}(M,E)=0\}$. For instance, if P is a prime ideal of R, then the collection of P-torsion modules are precisely those modules cogenerated by $E(R/P)$. A seminoetherian ring R is stable if and only if $E(R/P)$ cogenerates a stable localizing subcategory for all prime ideals P in R [3, Proposition 20.12]. Our next lemma gives a criteria for when such a subcategory is not stable.

Lemma 2.3. *Let P be a prime ideal in the seminoetherian ring R. The localizing subcategory cogenerated by $E(R/P)$ is not stable if and only if there exists a finitely generated module H such that* $\mathrm{Hom}(H,E(R/P))\neq 0$ *yet for all $Q\in \mathrm{ass}(H)$, $Q\not\subseteq P$.*

Proof. Assume the localizing subcategory cogenerated by $E(R/P)$ is not stable. Clearly there exists an injective module E and a homomorphism $f:E\to E(R/P)$ where $f\neq 0$ yet for all $Q\in \mathrm{ass}(E)$, $Q\not\subseteq P$. So let H be a finitely generated submodule of E which is not killed by f.

The converse is clear since we are working over a seminoetherian ring. □

We should note one fact which is rather obvious from the definition. If R is a stable ring and $Q(R)$ the ring of quotients of R with respect to a multiplicative set, then $Q(R)$ is also stable.

Theorem 2.4. *Let S be a subidealizer of $\mathcal{M}$ in T where $\mathcal{M}$ is a maximal ideal of S. If T is noetherian, then S is stable.*

Proof. Since T is noetherian it follows from [5, Satz 3.1] that S is seminoetherian. Thus it suffices to show that every localizing subcategory cogenerated by an indecomposable injective module is stable.

Let E be an indecomposable injective S-module such that $\mathrm{ass}(E)=\{P\}$ where $P\neq\mathcal{M}$. Let N be an S-module such that $\mathrm{Hom}(N,E)=0$. Assume there exists $f:E(N)\to E$ which is not zero. Notice that $\mathrm{ann}_E(\mathcal{M})=0$, so by Lemma 2.2 there exists a nonzero T-homomorphism from $E(N)\mathcal{M}$ to $E\mathcal{M}$. Since T is noetherian there exists $Q\in \mathrm{ass}_T(E(N)\mathcal{M})$ and $P\in \mathrm{ass}_T(E\mathcal{M})$ such that $Q\subseteq P$. However, $Q\cap S\in \mathrm{ass}_S(E(N))$ and $P\cap S\in \mathrm{ass}_S(E\mathcal{M})$. Since $Q\cap S\subseteq P\cap S$ we have a contradiction because $\mathrm{ass}_S(E(N))=\mathrm{ass}_S(N)$. So E cogenerates a stable torsion theory.

Now let $E=E_S(S/\mathcal{M})$. Assume E does not cogenerate a stable torsion theory. By Lemma 2.3 there exists a finitely generated S-module H such that $\mathrm{Hom}(H,E)\neq 0$ yet for all $Q\in \mathrm{ass}(H)$, $Q\not\subseteq \mathcal{M}$. Notice $H\mathcal{M}$ is a finitely generated T-module so $\mathrm{ass}_T(H\mathcal{M})$ is finite (after all T is noetherian). Also note that if $P\in \mathrm{ass}_T(H\mathcal{M})$, then $\mathcal{M}\not\subseteq P$ otherwise $\mathcal{M}=P\cap S\in \mathrm{ass}(H)$ a contradiction. So by our discussion prior to Lemma 2.2 $E_T(H\mathcal{M})$ is injective as an S-module. Since $\mathrm{ann}_H(\mathcal{M})=0$ (otherwise $\mathcal{M}\in \mathrm{ass}(H)$) $H\mathcal{M}$ is essential in H. Thus $E_S(H)=E_S(H\mathcal{M})\subseteq E_T(H\mathcal{M})$. Let $\{P_1,P_2,\ldots,P_n\}=\mathrm{ass}_T(H\mathcal{M})$. Since T is noetherian, every element of $E_T(H\mathcal{M})$ and hence $E_S(H)$ is killed by $(P_1 \cdot P_2 \cdot P_3 \cdots P_n)^r$ for some positive integer r. Now let $f: H \to E$ be nonzero. Then there exists $h \in H$ such that $f(h) \neq 0$ and $f(h)\mathcal{M} = 0$. Thus $f(h)$ is annihilated by $\mathcal{M}$ and $(p_1 \cdot p_2 \cdots p_n)^r$ where $p_i = P_i \cap S$. Clearly $p_i \in \mathrm{ass}_S(H)$ so $p_i \not\subseteq \mathcal{M}$. Hence $(p_1 \cdot p_2 \cdots p_n)^r + \mathcal{M} = S$, an impossibility. Thus H cannot exist. Therefore S is stable.

Example 2. Let R be a noetherian ring and let P be a maximal ideal of R. Then $R + P[x] \not\subseteq R[x]$ satisfies the relationship of S and T in Theorem 2.4 with $\mathcal{M} = P[x]$. So $R + P[x]$ is stable. However it is not noetherian if P is not idempotent, and it is not perfect as long as R is not perfect.

Example 3. Let M be a finitely generated abelian monoid. Let N be an ideal of M which is not finitely generated as a semigroup (e.g. let M be the free abelian monoid on $\{x, y\}$ and N the ideal generated by the set $\{x, xy, xy^2 \cdots\}$). N^1 will denote the submonoid of M obtained by adjoining 1 to N. Then, for any field F, the monoid ring FN^1 and FM satisfy the relationship of Theorem 2.4 where $\mathcal{M}$ is the ideal FN. So FN^1 is stable yet it is not noetherian since N is infinitely generated.

Example 4. Let F be an infinite dimensional field extension of a field K. If $T = F[[x]]$ and $S = K + \mathcal{M}$ where $\mathcal{M}$ is the unique maximal ideal of T, we are again in the situation of Theorem 2.4, yet S is not noetherian.

3. When stable implies noetherian

In [2] it was shown that 'stability' implies 'noetherian' for certain classes of commutative rings. In this section we present more results along those lines. In particular we show that if $R[x]$ is stable, then R is noetherian. We also prove that a stable ring with certain properties (satisfied by most of the examples presented in Section 2) that has either finite global dimension or is coherent must be noetherian.

Lemma 3.1. *Let R be a stable ring which is not noetherian. Then there exists a maximal ideal P of R such that P_P/P_P^2 is infinitely generated over R_P.*

Proof. By [2, Proposition 8] there exists a maximal ideal P such that R_P is not noetherian. So we can assume R is a stable local ring with maximal ideal P such that R is not noetherian. Let E be the injective envelope of the simple R module. We will show that if P/P^2 is finitely generated, then E is artinian. By virtue of [11, Theorem 2], this implies R is noetherian, a contradiction.

Assume P/P^2 is finitely generated over R/P^2. It follows by induction that P/P^n is finitely generated over R/P^n for any integer n. Hence R/P^n is artinian. Thus $\mathrm{soc}_n(E) = \mathrm{ann}_E(P^n)$, which is the injective envelope of the unique simple R/P^n-

module is also artinian [10, 4.30]. Now let $x \in E$ and let $I = \mathrm{ann}_R(x)$. By Proposition 1.4, E is semiartinian, so R/I is a perfect ring. So P/P^2 finitely generated implies that R/I is artinian [7, Lemma 11]. Hence $E = \bigcup_{n=1}^{\infty} \mathrm{soc}_n(E)$. Therefore $E/\mathrm{soc}_1(E)$ has an essential, finite dimensional socle. The proof that E is artinian now follows as in [2, Proposition 12]. □

Theorem 3.2. *If the polynomial ring $R[x]$ is stable, then R is noetherian.*

Proof. Clearly, by Proposition 1.1 R is stable. Suppose R is not noetherian. By Lemma 3.1 we can assume R is a local, stable ring with maximal ideal P such that P/P^2 is not a finitely generated module. Therefore, we lose nothing if we also assume that R is a stable ring with a unique, infinitely generated prime ideal P such that $P^2 = 0$.

Let $H = R/P$, then H is a simple $R[x]$-module. If $R[x]$ is stable, $E_{R[x]}(H)$ should be semiartinian by Proposition 1.2. We will show that this is not the case.

Since P is infinitely generated and $P^2 = 0$, it follows that $P \simeq \bigoplus_I R/P$ where I is infinite. So $P[x] \simeq \bigoplus_I R/P[x]$. Let E be the injective null of H over T where $T = R/P[x]$. From the exact sequence

$$0 \to P[x] \to R[x] \to T \to 0$$

we obtain

$$\begin{array}{l} 0 \to \mathrm{Hom}_T(T, E) \to \mathrm{Hom}_T(R[x], E] \to \mathrm{Hom}_T(P[x], E) \to 0 \\ \qquad\quad \wr\wr \\ \qquad\quad E \end{array}$$

Observe that the middle term of the above sequence is an injective $R[x]$-module. We also claim that it is essential extension (as an $R[x]$-module) of $\mathrm{Hom}_T(T, E)$. For if φ is in the former but not in the latter, then $\varphi(q) \neq 0$ for some $q \in P[x]$. Thus $0 \neq q \cdot \varphi \in \mathrm{Hom}_T(T, E)$ since $P[x]^2 = 0$. Therefore $\mathrm{Hom}_T(R[x], E) = E_{R[x]}(H)$. On the other hand

$$\mathrm{Hom}_T(P[x], E) \simeq \mathrm{Hom}_T\left(\bigoplus_I T, E\right) \simeq \prod_I \mathrm{Hom}_T(T, E) \simeq \prod_I E.$$

Since T is a PID, the infinite product must contain a torsion free T-module. Hence it is not semiartinian as an $R[x]$-module. Since it is a homomorphic image of $E_{R[x]}(H)$, we have a contradiction. □

Now we will start working towards Theorem 3.5 which states that a stable ring with finite global dimension and certain other properties has to be noetherian.

We will denote the projective (injective) dimension of the module M by $\mathrm{pd}(M)$ ($\mathrm{id}(M)$ respectively). The global dimension of the ring R will be denoted $\mathrm{gl\,dim}\, R$.

Let R be any local ring with simple module H. Let M be an arbitrary R-module.

We have an injective resolution

$$0 \to M \to E_0 \xrightarrow{d_0} E_1 \to \cdots - E_n \xrightarrow{d_n} \cdots$$

Then $\operatorname{Ext}^n(H, M) \simeq \operatorname{soc}(E_n)$. Now if we assume R is stable and $M = H$, then a simple induction argument shows that $\operatorname{soc}(E_n) = 0$ only if $E_n = 0$. Thus we have proven the following.

Lemma 3.3. *Let R be a stable local ring with simple module H. If* $\operatorname{Ext}^n(H, H) = 0$, *then* $\operatorname{id}(H) < n$.

Proposition 3.4. *A stable seminoetherian ring with finite global dimension is semi-prime.*

Proof. Assume the nilradical N is not zero. Since R is seminoetherian there exists $a \in N$ such that $\operatorname{ann}(a) = P$, a prime ideal.

The ring R_P has finite global dimension and contains the simple nilpotent ideal $(Ra)_P$. Let $n = \operatorname{id}_{R_P}((R_a)_P)$. Since the ideal $(Ra)_P$ is not a direct summand of R_P, $n > 0$. Consider the short exact sequence

$$0 \to (Ra)_P \to R_P \to (R/Ra)_P \to 0.$$

From which we obtain the exact sequence

$$\cdots \to \operatorname{Ext}^n_{R_P}(R_P, (Ra)_P) \to \operatorname{Ext}^n_{R_P}((Ra)_P, (Ra)_P) \to \operatorname{Ext}^{n+1}_{R_P}((R/Ra)_P, (Ra)_P) \to \cdots$$

Clearly the modules on each of the sequence are zero, yet by Lemma 3.3 the middle term is not zero, quite an impossibility. So $N = 0$.

Definition. A ring R is *G-stable* if R is stable and R/N has finite Goldie dimension for every semiprime ideal N. This is equivalent to saying every semiprime ideal is the intersection of a finite number of prime ideals.

Recall that G-dim(M) stands for the Gabirel dimension of the module M. Assume R is seminoetherian. It was shown in [4, Proposition 3.3] that if R is a domain, then R is Gabriel simple, i.e., every factor module has Gabriel dimension less than R. In any case

$$\text{G-dim}(R) = \sup\{\text{G-dim}(R/P) \mid P \text{ a prime ideal of } R\}$$

[4, Lemma 3.1]. So for any seminoetherian ring R, given an ideal I such that I is not in any minimal prime ideal $\text{G-dim}(R/I) < \text{G-dim}(R)$. Also note that if G-dim(R) is finite and if Q is a prime ideal which is not maximal, then it follows from the definition that $\text{G-dim}(R_Q) < \text{G-dim}(R)$.

Theorem 3.5. *If R is a G-stable ring with finite global dimension and finite Gabriel dimension, then R is noetherian.*

Proof. By [2, Proposition 8] we can assume R is local with unique maximal P. We proceed with induction on G-dim(R). If G-dim(R) = 0, then every R-module has an essential socle. Hence by the result of Auslander [1], $\mathrm{pd}(M) \le \mathrm{pd}(R/P)$ for any R-module M. Thus $\mathrm{gl\,dim}\, R = \mathrm{pd}(R/P)$. However R contains a copy of R/P, so $\mathrm{gl\,dim}\, R = 0$.

Now assume the theorem is true for Gabriel dimension less than k and suppose G-dim(R) = k. By Proposition 3.5, R is semiprime. let $P_1, P_2, \ldots, P_n$ denote the minimal prime ideals of R. By Lemma 3.1, $P \neq P^2$. So there exists $a \in P - P^2$ (in P but not in P^2) which is a non-zero divisor. If not, $P - P^2 \subseteq P_1 \cup P_2 \cup \cdots \cup P_n$ which by [6, Theorem 83] implies $P \subseteq P_i$ for some i, a contradiction.

Consider the ring $T = R/Ra$. Note that T is G-stable and since G-dim(R) is finite G-dim(T) < k. We claim $\mathrm{pd}_T(R/P) < \infty$. Clearly it suffices to show that $\mathrm{pd}_T(P/Ra)$ is finite. By [6, Theorem D, p. 124] we know that $\mathrm{pd}_T(P/Pa)$ is finite. Now consider the exact sequence

$$0 \to Ra/Pa \to P/Pa \to P/Ra \to 0.$$

Since R is local and $a \notin P^2$, $Ra \cap P^2 = Pa$. So Ra/Pa is isomorphic to a submodule of P/P^2. Since P/P^2 is semisimple there exists a submodule U of P such that $Ra + U = P$ and $Ra \cap U \cap P^2$. Then

$$Ra \cap U = Pa \cap U \cap P^2 \subseteq Ra \cap P^2 = Pa.$$

Thus Ra/Pa is a direct summand of P/Pa. So $\mathrm{pd}_T(P/Ra) < \infty$ and the claim is proved.

If Q is a prime ideal containing Ra such that $Q \neq P$, then G-dim(R_Q) < k. So by induction R_Q is noetherian. $R_Q/(Ra)_Q \simeq T_{\bar{Q}}$($\bar{Q}$ the image of Q in T) is of course noetherian and by [6, Theorem 161] has finite global dimension. Using the classical Krull dimension we also see that $\mathrm{gl\,dim}(T_{\bar{Q}}) < k$.

Let $l = \max(\mathrm{pd}_T(R/P), k-1)$. We claim T has global dimension at most l. Suppose N is a T-module with $\mathrm{id}_T(N) > l$. Take a T injective resolution of N.

$$0 \to N \to E_0 \to E_1 \to \cdots E_l \to E_{l+1} \to \cdots$$

Let Q be a prime ideal of R such that $R/Q \subseteq E_{l+1}$. By [2, Proposition 1] the exact sequence

$$0 \to N_{\bar{Q}} \to (E_0)_{\bar{Q}} \to (E_1)_{\bar{Q}} \to \cdots \to (E_{l+1})_{\bar{Q}}$$

is an injective resolution of $N_{\bar{Q}}$ over $T_{\bar{Q}}$. Clearly $(E_{l+1})_{\bar{Q}}$ has a nonzero socle (as a module over $T_{\bar{Q}}$). So

$$\mathrm{Ext}^{l+1}_{T_{\bar{Q}}}((R/Q)_{\bar{Q}}, N_{\bar{Q}}) \neq 0.$$

But $\mathrm{pd}_{T_{\bar{Q}}}[(R/Q)_{\bar{Q}}] \le l$, a contradiction. So $T = R/Ra$ has finite global dimension. Thus, by induction, it is noetherian. It follows that P is finitely generated. So by Lemma 3.1, R is noetherian.

Corollary 3.6. *The rings described in Examples 3 and 4 of Section 2 have infinite global dimension.*

Recall a ring R is called coherent if every finitely generated ideal is finitely presented. This implies that the annihilator of any finitely generated ideal is again finitely generated. We will use this to show that in a seminoetherian stable ring, coherence implies noetherian.

Proposition 3.7. *If R is stable, seminoetherian and coherent, then R is noetherian.*

Proof. We proceed by induction on G-dim(R). Again by [2, Proposition 8] we can assume R is local with maximal ideal P.

If G-dim$(R)=0$, then P is the only prime ideal of R. Thus, P is the annihilator of some element of R. Since R is coherent, P is finitely generated. So by Lemma 3.1, R is noetherian.

Suppose G-dim$(R)=\alpha$. Since R is seminoetherian there exists a prime ideal Q of R such that $Q=\mathrm{ann}(y)$, $y\in R$. So Q is finitely generated. Pick $x\notin Q$. Then $R/(Rx+Q)$ is stable, coherent and by [4, Proposition 3.3] has Gabriel dimension less than α. Thus by induction $R/(Rx+Q)$ is noetherian. So P is finitely generated and again we are done.

Remark. It follows from Proposition 3.7 that all the rings constructed using the techniques of Section 2 which are not noetherian also cannot be coherent.

We end with some comments on the conditions for R in Theorem 3.5. That R needs to be G-stable does not appear to be very restrictive since all the stable rings we have are also G-stable. It would be interesting to know if all stable rings are G-stable. The fact that R must have finite Gabriel dimension is more restrictive since it is easy to construct stable rings with infinite Gabriel dimension using the techniques of Section 2. However, we have no counter examples to show that this condition can't be dropped.

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